Goldstein

2.3. Prove 2 pts. have straight line between them as shortest distance in space.

We use minimal action principle, the lagrangian is

$$L = \int \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \quad \text{where } \dot{x} \text{ denotes } \frac{dx}{dt},$$

the curve is defined by $x(t), y(t), z(t)$.

The Euler Lagrange equation gives

$$\frac{d}{dt}\left[\frac{\dot{x}}{L}\right] = 0 \Rightarrow \frac{\dot{x}}{L} \text{ is constant along the curve.}$$

Since $\frac{\dot{x}}{L} = \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}$, for this quantity to be a constant along the curve, we must have $\dot{y} = \alpha \dot{x}$, $\dot{z} = \beta \dot{x}$ for some constant α, β. This allows the Lagrangian to be rewritten as $L = \dot{x}\sqrt{1 + \alpha^2 + \beta^2}$.

Applying Euler-Lagrange eq. gives $\ddot{x} = 0$, which means x is first order in t. By symmetry, y, z are also 1st order in t. Thus $x(t), y(t), z(t)$ is a straight line.